

# Argonne National Laboratory

## ANALOG COMPUTATION OF TEMPERATURE DISTRIBUTION IN SOLIDS WITH ELECTRICAL HEAT-GENERATION AND TEMPERATURE-DEPENDENT PROPERTIES

by

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## NOMENCLATURE

a	scale factor for length, v/in.
b	scale factor for temperature, v/°F
k	thermal conductivity, Btu/(hr)(in.)(°F)
$\dot{q}'''$	time rate of internal heat generation, Btu/(hr)(in. <sup>3</sup> )
r	radius, in.
t	temperature, °F
A <sub>Ht</sub>	area normal to heat flux, in. <sup>2</sup> /in.
A <sub>cs</sub>	cross-sectional area normal to current flow
E	voltage gradient, v/in.
L	linear length, in.
V	volume of tube wall, in. <sup>3</sup> /in.

### Greek Letters

$\alpha$	temperature coefficient of electrical resistivity, ( $\mu\text{ohm}$ )(in.)/°F
$\gamma$	temperature coefficient of thermal conductivity, Btu/(hr)(in.)(°F <sup>2</sup> )
$\rho$	electrical resistivity, ( $\mu\text{ohm}$ )(in.)
$\tau$	machine variable representing time

### Subscripts

0	evaluated at 0°F
1	inside radius
2	outside radius
t	indicates temperature dependence

### Superscripts

'	indicates machine variable
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# ANALOG COMPUTATION OF TEMPERATURE DISTRIBUTION IN SOLIDS WITH ELECTRICAL HEAT-GENERATION AND TEMPERATURE-DEPENDENT PROPERTIES

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## ABSTRACT

A problem which frequently arises in experimental heat transfer work is that of determining the surface temperature of a tube in which heat is generated electrically. Solution of this problem involves a temperature measurement of the opposite surface to which a correction factor, the temperature drop through the tube wall, must be applied. This temperature drop is obtained through the solution of the differential equation governing the temperature distribution in the tube wall; however, in the case of temperature-dependent properties of thermal conductivity and electrical resistivity, the governing equation is nonlinear, which necessitates special solutions.

In this study a hypothetical surface-temperature problem is established, and the solution of the governing nonlinear differential equation is accomplished by means of an electronic analog computer. Assuming variable properties, the example used in this study is that of a one-dimensional steady-state heat flow through both a thick- and a thin-walled tube.

## I. INTRODUCTION

A problem which frequently arises in experimental heat transfer work is that of determining the surface temperature of a tube in which heat is generated electrically. Solution of this problem involves a temperature measurement of the opposite surface to which a correction factor, the temperature drop through the tube wall, must be applied. This temperature drop is obtained through the solution of the differential equation governing the temperature distribution in the tube wall; however, in the case of temperature-dependent properties of thermal conductivity and electrical resistivity, the governing equation is nonlinear, which necessitates special solutions.





Jakob<sup>(1)</sup> presented an analytical solution for the case of a solid cylinder in which the thermal conductivity is independent of temperature and the electrical resistivity a linear function of temperature.

Kreith and Summerfield<sup>(2)</sup> utilized a series solution, assuming that the thermal conductivity and electrical resistivity were linear functions of temperature, from which was obtained the overall temperature change across a tube wall as a function of an infinite series in ascending powers of wall thickness.

Dickinson and Welch<sup>(3)</sup> in an application of the Kreith-Summerfield solution to a thick-walled tube found it necessary to compute additional terms to assure convergence of the series solution.

Clark<sup>(4)</sup> simplified the Kreith-Summerfield solution by showing that the series solution could be written as the sum of two functions, the first being a geometric function (solution for constant properties) and the second a function which accounted for variations of thermal properties. The utility of this approach is that the influence of variation of the thermal property on the temperature drop through the solid can be computed quite easily; however, in the series solution, the uncertainties concerning convergence of the series, when temperature dependence is important, is still in question.

Stein and Gutstein<sup>(5)</sup> approached the solution of the differential equation by introducing a parameter which gives the effect of the influence of the temperature dependence of electrical conductivity on the temperature and heat-flux distributions through the solid. They then determined the percentage error in the calculated temperature drop which would result from neglecting the temperature dependence of electrical conductivity as a function of the parameter. The generalization was made that, if the temperature drop is computed from the surface heat flux and if the temperature dependence of electrical conductivity is neglected, the resulting error in the temperature drop will be less than 5%, even though the electrical conductivity may change by 25%. For most metals, a change in electrical conductivity of this magnitude requires temperature changes of the order of hundreds of degrees. It was concluded that only in unusual cases would it be necessary to account for the temperature dependence of electrical conductivity. For the unusual case, an incremental procedure for calculating the temperature distribution was presented, which is based on exact integrations of the heat-conduction equation over small increments of solid thicknesses in which it is assumed that the temperature dependence of electrical conductivity is negligible.

More recently, the series solution<sup>(4)</sup> was reviewed by Bergles and Rohsenow.<sup>(6)</sup> They discuss the use of a guard heater to insure an adiabatic outer surface. The results from the series solution presented show a



larger temperature drop for higher surface temperature. It appears that these results do not agree with previously published results or with the results of this study.

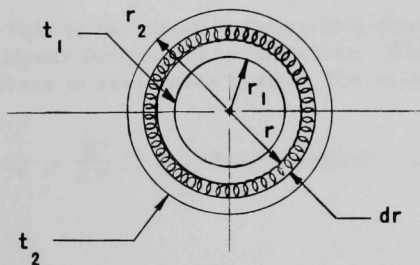
### Analog Computer Solutions

The solution of the nonlinear differential equation by the use of an analog computer eliminates the uncertainties concerning convergence of a series solution. With the use of the analog computer, the approximations made in the Stein-Gutstein<sup>(5)</sup> incremental method would not be necessary. Moreover, the effect of a nonadiabatic boundary condition can be studied quite easily with the analog, since this is one of the initial conditions of the problem.

## II. EQUATION DEVELOPMENT

The flow of heat in a hollow cylinder (Fig. 1), long enough so that end heat losses can be considered negligible, is one dimensional. In Fig. 1 the heat flux entering the shaded volume element  $dV$  is  $-k_t A (dt/dr)$ ; the heat generation in the volume is  $\dot{q}''' dV$ ; and the rate of heat transfer at  $r + dr$  is

$$\left( -k_t A \frac{dt}{dr} \right) + \frac{\partial}{\partial r} \left( -k_t A \frac{dt}{dr} \right) dr$$



### BOUNDARY CONDITIONS

$$\left. \frac{dt}{dr} \right|_{r=r_2} = 0 \quad t = t_2$$

$$\left. \frac{dt}{dr} \right|_{r=r_1} = 0 \quad t = t_1$$

Fig. 1. Cross Section of Heater Tube with Boundary Conditions

The energy balance becomes

$$\left( -k_t A \frac{dt}{dr} \right) + \dot{q}''' A dr = \left( -k_t A \frac{dt}{dr} \right) + \frac{\partial}{\partial r} \left( -k_t A \frac{dt}{dr} \right) dr \quad (1)$$





In Eq. (1) substituting  $A = 2\pi r$  per unit length and dropping the partial derivative notation, since  $t$  is a function of  $r$  only, one obtains

$$\frac{1}{r} \frac{d}{dr} \left( k_t r \frac{dt}{dr} \right) + \dot{q}_t''' = 0 \quad . \quad (2)$$

Performing the indicated differentiation, Eq. (2) becomes

$$\frac{1}{r} \left[ k_t r \left( \frac{d^2 t}{dr^2} \right) + k_t \left( \frac{dt}{dr} \right) + r \left( \frac{dt}{dr} \right) \left( \frac{dk_t}{dr} \right) \right] + \dot{q}_t''' = 0 \quad . \quad (3)$$

At this point it is necessary to specify the relationship between  $k$  and  $t$ . If a linear variation of the form

$$k_t = k_0 [1 + \gamma t] \quad (4)$$

is specified, then

$$\frac{dk_t}{dr} = \left( \frac{dk_t}{dt} \right) \left( \frac{dt}{dr} \right) = k_0 \gamma \left( \frac{dt}{dr} \right) \quad . \quad (5)$$

Substituting Eq. (5) into Eq. (3) gives

$$\frac{d^2 t}{dr^2} + \frac{1}{r} \left( \frac{dt}{dr} \right) + \frac{k_0 \gamma}{k_t} \left( \frac{dt}{dr} \right)^2 + \frac{\dot{q}_t'''}{k_t} = 0 \quad . \quad (6)$$

This is the equation describing the temperature distribution when  $k_t$  is a linear function of temperature. When the internal heat generation is by means of resistance heating, the volumetric heat generation is given by

$$\dot{q}_t''' = \frac{E^2}{RV} (3.413) \text{ Btu}/(\text{watt})(\text{hr}) \quad , \quad (7)$$

where

$$R = \rho L / A_{CS} \quad .$$

If the electrical resistivity  $\rho$  is a linear function of  $t$ , for example

$$\rho = \rho_0 [1 + \alpha t] \quad ,$$

then substituting for  $\rho$  in Eq. (7) and for  $\dot{q}_t'''$  in Eq. (6) results in

$$\frac{d^2 t}{dr^2} + \frac{1}{r} \left( \frac{dt}{dr} \right) + \frac{\gamma}{(1 + \gamma t)} \left( \frac{dt}{dr} \right)^2 + \frac{E^2 (3.413)}{[\rho_0 (1 + \alpha t) L / A_{CS}] V k_0 (1 + \gamma t)} = 0 \quad . \quad (8)$$



Since

$$V = A_{cs}L$$

and

$$L = \text{unit length} \quad ,$$

Eq. (8) can be reduced to

$$\frac{d^2t}{dr^2} + \frac{1}{r} \left( \frac{dt}{dr} \right) + \frac{\gamma}{(1 + \gamma t)} \left( \frac{dt}{dr} \right)^2 + \frac{E^2(3.413)}{\rho_0(1 + \alpha t)k_0(1 + \gamma t)} = 0 \quad . \quad (9)$$

If  $k$  and  $\rho$  are considered constant and evaluated at some reference temperature, then Eq. (2) can be written as

$$\frac{d^2t}{dr^2} + \frac{1}{r} \left( \frac{dt}{dr} \right) + \frac{E^2(3.413)}{\rho k} = 0 \quad . \quad (10)$$

### III. ANALOG EQUATION TRANSFORMATION

The transformation of Eqs. (9) and (10) into a form suitable for analog representation is made by the following substitutions:

$$r = \tau; \quad \tau' = a\tau; \quad t' = bt \quad .$$

Thus,

$$\frac{dt}{dr} = \frac{d(t'/b)}{d(\tau'/a)} = \frac{a}{b} \left( \frac{dt'}{d\tau'} \right)$$

and

$$\frac{d^2t}{dr^2} = \frac{d}{d(\tau'/a)} \left[ \frac{a}{b} \frac{dt'}{d\tau'} \right] = \left( \frac{a^2}{b} \right) \left( \frac{d^2t'}{d\tau'^2} \right) \quad .$$

Making the substitution into Eqs. (9) and (10) and multiplying by  $b/a^2$ , one obtains

$$\frac{d^2t'}{d\tau'^2} + \frac{1}{\tau'} \left( \frac{dt'}{d\tau'} \right) + \frac{\gamma}{[1 + (\gamma/b)t']b} \left( \frac{dt'}{d\tau'} \right)^2 + \frac{b}{a^2} \left( \frac{E^2(3.413)}{\rho_0[1 + (\alpha t'/b)]k_0[1 + (\gamma t'/b)]} \right) = 0 \quad (11)$$

and

$$\frac{d^2t'}{d\tau'^2} + \frac{1}{\tau'} \left( \frac{dt'}{d\tau'} \right) + \frac{b}{a^2} \left( \frac{E^2(3.413)}{\rho k} \right) = 0 \quad . \quad (12)$$





To assign the scale factors it is necessary to determine the maximum ranges of the variables. The maximum temperature expected is 600°F. The maximum tube size is 0.50 in. radius. Then,

$$t'(0) = bt(0); \quad 100 \text{ v} = b(600^\circ\text{F}); \quad b = 1/10 \text{ v}/^\circ\text{F} \quad ,$$

$$\text{and} \quad \tau'(0) = a\tau(0); \quad 100 \text{ v} = a(0.50 \text{ in.}); \quad a = 1 \times 10^2 \text{ v}/\text{in.}$$

Introducing these values of  $a$  and  $b$  into Eqs. (11) and (12) and introducing values<sup>(3)</sup> of electrical resistivity and thermal conductivity from Fig. 2,

$$\begin{aligned} \frac{d^2 t'}{d\tau'^2} = & -\frac{1}{\tau'} \left( \frac{dt'}{d\tau'} \right) - \frac{(0.00517)}{(1 + 0.00517t')} \left( \frac{dt'}{d\tau'} \right)^2 \\ & - \frac{1}{10^5} \left[ \frac{E^2(3.413)}{\rho_0(1 + 0.0062t')k_0(1 + 0.00517t')} \right] \end{aligned} \quad (13)$$

and

$$\frac{d^2 t'}{d\tau'^2} = -\frac{1}{\tau'} \left( \frac{dt'}{d\tau'} \right) - \frac{1}{10^5} \left[ \frac{E^2(3.413)}{\rho k} \right] \quad (14)$$

The analog circuit diagrams for Eqs. (13) and (14) are shown in Figs. 3 and 4. The potentiometer settings for these figures are listed in Tables 1 and 2.

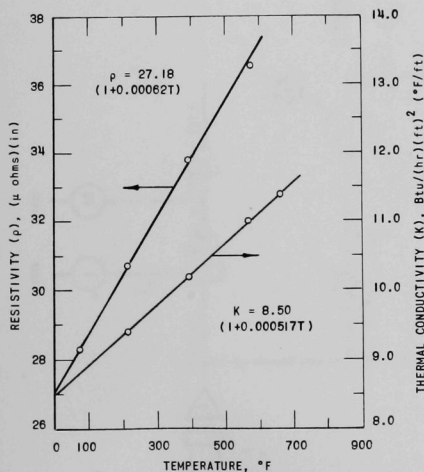


Fig. 2

Thermal Conductivity and Electrical Resistivity Versus Temperature for Type 304 Stainless Steel



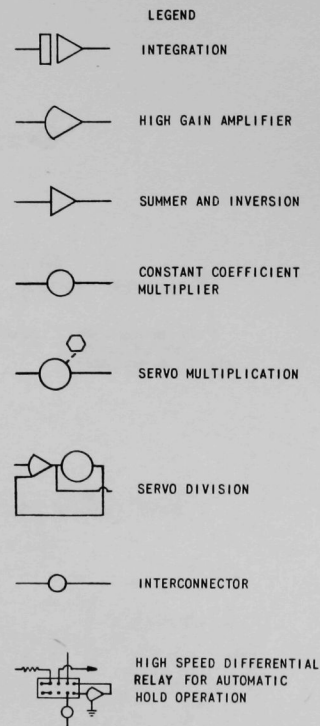
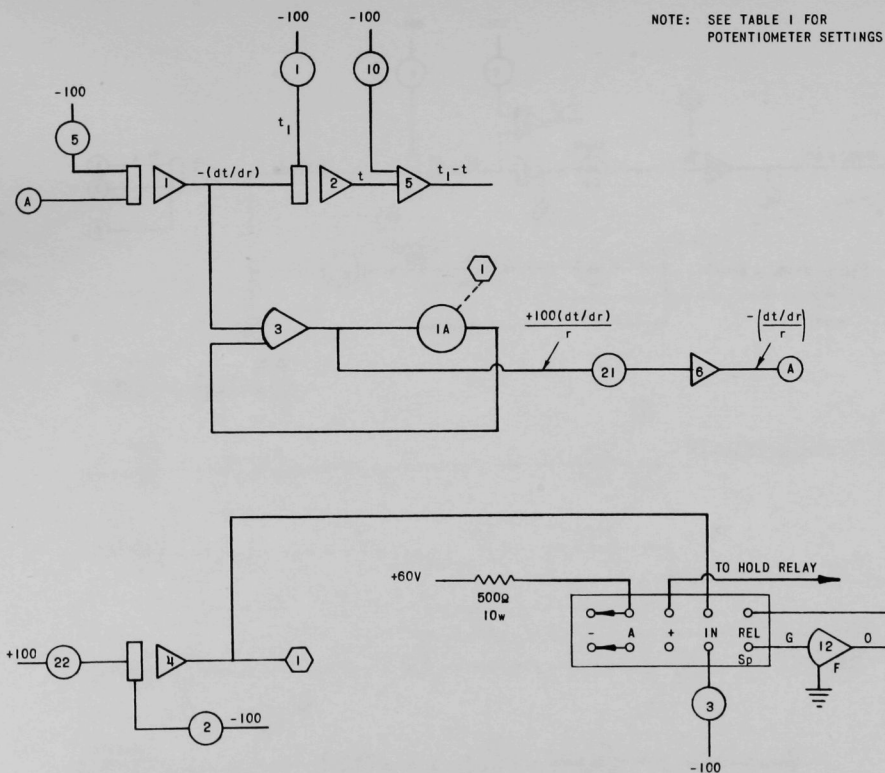


Fig. 3. Analog Diagram of Solution for Case of Constant Thermal and Electrical Conductivity



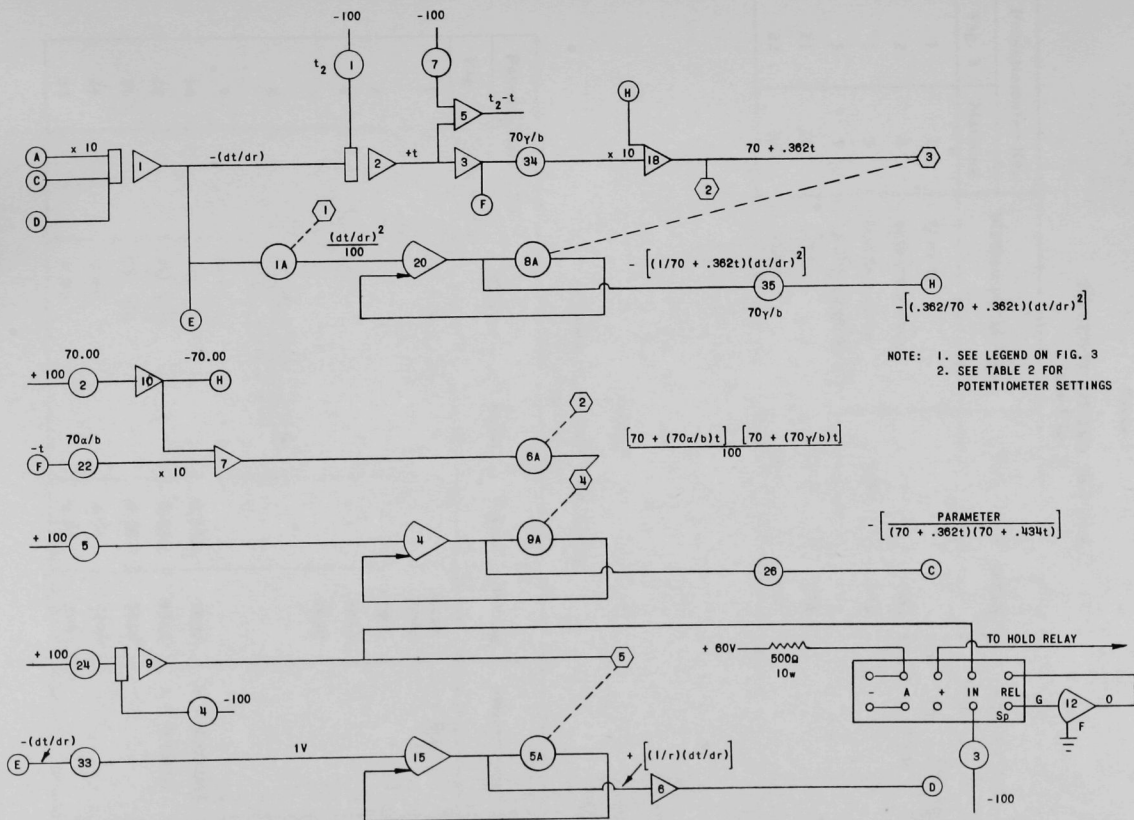


Fig. 4. Analog Diagram of Solution for Case of Temperature Dependent Thermal and Electrical Conductivity



Table 1

POTENTIOMETER SETTINGS  
(see Fig. 3)

Potentiometer No.		Mathematical Value	Value	Setting	Parameters
Fig. 3	Machine				
1	1	$t_1$ or $t_2$			$t' = bt; \quad t \leq 1000$ $b = 0.1 \sqrt{^\circ F}$
2	2	outside radius		-5000	
3	3	inside radius	4650	-4650	
5	5	$(3.413/\rho k)E^2(b/a^2)$	parameter		
21	21	1/100	0.01	0100	
22	22	1/100	0.01	0100	

Table 2

POTENTIOMETER SETTINGS  
(see Fig. 4)

Potentiometer No.		Mathematical Value	Value	Setting	Parameters
Fig. 4	Machine				
1		$t_1$ or $t_2 = 600$ $= 300$		-6000 -3000	$b = 0.1$
2				+7000	
4		$r_2$		-5000	
3		$r_1$		-4650	
5		$\frac{b}{a^2} \left[ \frac{70 \times 70 \times 3.413 E^2}{(\rho k)_0 \times 10^{-6}} \right]$			
7		$t_1$ or $t_2$			$\gamma = 0.000517$ $\alpha = 0.00062$
34		$70 \sqrt{b}$	0.362	3620	
22		$70 \alpha/b$	0.434	4340	
35		$70 \sqrt{b}$	0.362	3620	
24		0.01	0.01	0100	
33		0.01	0.01	0100	





#### IV. DISCUSSION OF RESULTS

The results of the computer solutions for the cases of heat flux through the inner and outer surfaces of both thick- and thin-walled 1-in.-OD tubing are shown in Figs. 5 and 6. Figure 5 shows that the effect of variable properties on the thin-walled tube studied does not become important until the temperature drop through the tube wall reaches or exceeds approximately 70°F. Comparison of the curves of Fig. 5 indicates that the higher the initial wall temperature, the lower the temperature drop through the wall. Figure 6 shows the results for the thick-walled tube. It should be noted that the effect of initial surface temperature has a much greater effect on the temperature drop for the thick-walled tube than for the thin-walled tube. Figure 6 (for  $t_1 = 300^\circ\text{F}$ ) also shows that an adiabatic inside or outside tube wall has only a negligible effect on the temperature drop across the tube wall.

In applying the series solution to this problem, Clark<sup>(4)</sup> showed that, when the variation of properties could be neglected, the temperature drop was a linear function of the heat flux. This is shown in Figs. 5 and 6 along with the variable-property solution. It is evident from examining Figs. 5 and 6 that the effect of variation of the thermal property becomes more pronounced in the thick-walled tube with the attendant greater temperature drop.

#### V. CONCLUSIONS

This study has shown the analog solution of the temperature distribution in an electrically heated tube to be flexible in its application. The question of convergence in the series solution is not encountered with the analog solution. By means of the analog solution the boundary conditions at the two surfaces can be changed quite easily and, in particular, the influence of a nonadiabatic boundary can be investigated.

When the assumption of constant properties can be made, it is recommended that Clark's<sup>(4)</sup> constant-property solution be used to calculate the temperature drop at one heat flux. Then, since the temperature drop is a linear function of the heat flux, all other temperatures can be read from a plot of the temperature drop versus the heat flux.



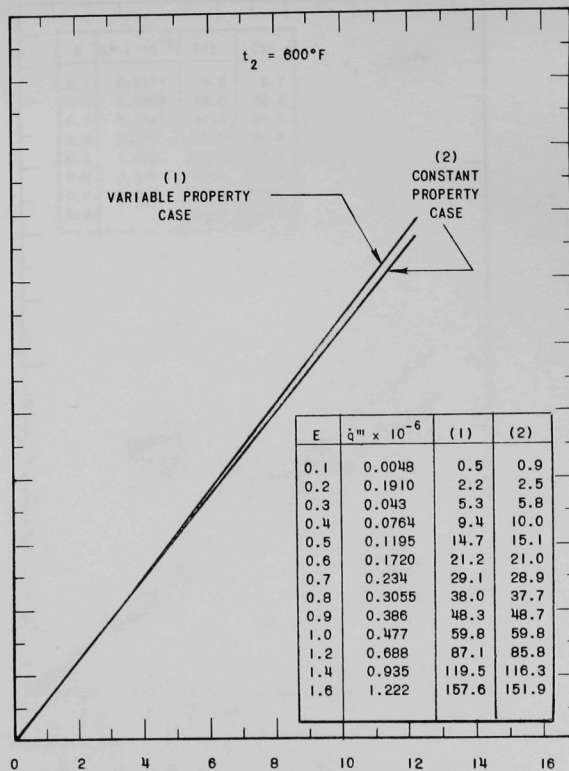
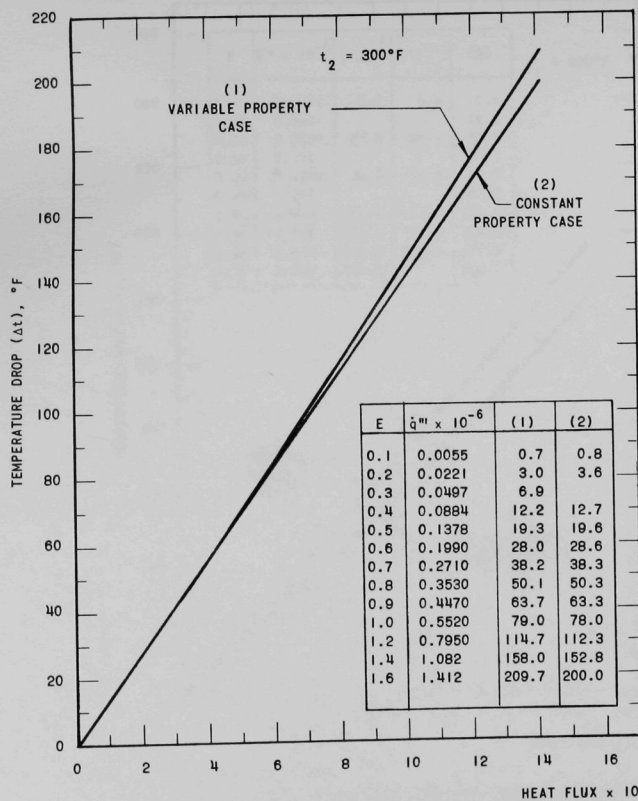


Fig. 5. Comparison of the Temperature Drop across a Thin-walled Tube for the Constant and Variable Property Cases at  $t = t_2$  of  $300^\circ\text{F}$  and  $600^\circ\text{F}$



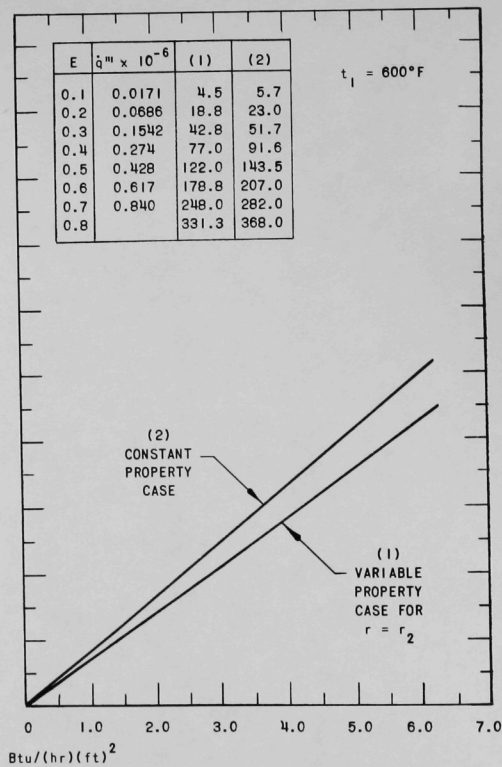
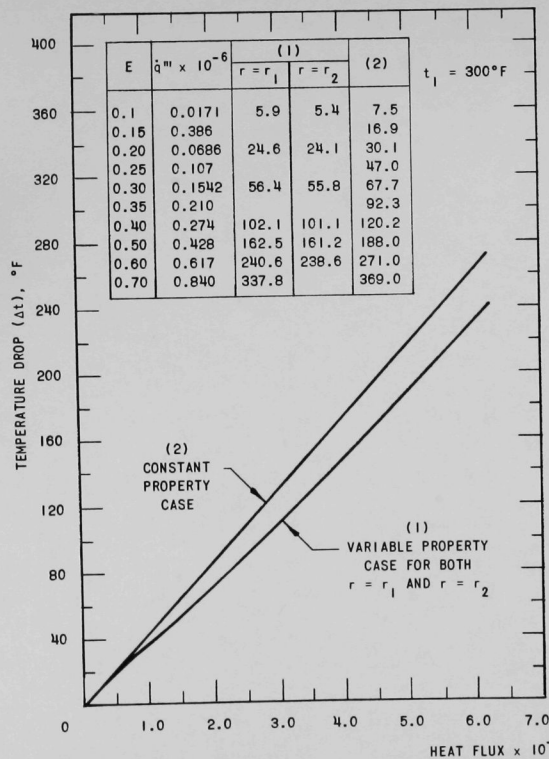


Fig. 6. Comparison of the Temperature Drop across a Thick-walled Tube for the Constant and Variable Property Cases at  $t = t_1$  of  $300^\circ\text{F}$  and  $600^\circ\text{F}$



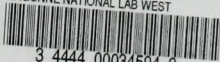


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